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Quaternionic Hilbert space and colour confinement: II. The admissible symmetry groups

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Abstract. The classification problem of the admissible (with respect to the quaternionic structure of the Hilbert space) representations of the semi-simple compact Lie groups is considered. It is found that a symmetry group G must be of the form $G = G_F \times G_c$ where the colour group G_c is isomorphic to the SU($3r$) and r is odd. The natural selection rules generated by quaternionic structure are equivalent to the confinement of colour, i.e. total algebraic confinement of $SU(3r)_c$ degrees of freedom holds.

1. Introduction

In the previous paper (Rembieliński 1979) (hereafter cited as paper I) it was shown that the formalism of the quaternionic Hilbert space (QHS) with complex geometry can be adequate for the description of the coloured hadron states. The results of I can be summarised as follows:

(a) The QHS with complex geometry is isomorphic to the complex Hilbert space (CHS) with appropriate structure essentially determined by the representations **1**, **2**, and $\bar{\mathbf{2}}$ of the unitary group $U(2)_c$. If the theory possesses a symmetry group G then $U(2)_c \subset G$. The admissible representations D(G) of G contain only the representations **1**, **2** and $\bar{\mathbf{2}}$ of $U(2)_c$ i.e. $D(G) \downarrow U(2)_c = (\oplus \mathbf{1}) \oplus (\oplus \mathbf{2}) \oplus (\oplus \bar{\mathbf{2}})$.

(b) This result allows us to define in a unique and consistent manner the 'tensor' product of the QHS:

$$\mathscr{H}_1 \times \mathscr{H}_2 \times \ldots \times \mathscr{H}_n \doteqdot \Pi(\mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \ldots \otimes \mathscr{H}_n).$$

Here $\mathscr{H}_k$ are the carrier spaces of the admissible representations of G and Π projects the standard tensor product ($\otimes$) of $\mathscr{H}_k$ on the whole subspace of the admissible representations.

(c) The observable states must necessarily be singlets of the group $SU(2)_c \subset U(2)_c$. Thus the $SU(2)_c$ degrees of freedom can be interpreted as the colour, i.e. $SU(2)_c$ is a subgroup of the colour group G_c.

In this paper the classification problem of the admissible (by quaternionic structure) representations of the classical semi-simple compact Lie groups is considered. The plan of this article is as follows. In § 2 we give the branching rules for reduction of the simple† group G_c to the subgroup $SU(2)_c$. The representations admissible with respect

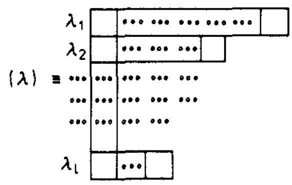
† It is sufficient to consider only simple groups. This follows from the fact that the condition $D(G) \downarrow SU(2)_c = (\oplus \mathbf{1}) \oplus (\oplus \mathbf{2})$ implies that $SU(2)_c$ must belong to a simple component of G. Thus the symmetry group G has the form $G = G_F \times G_c$ where the colour group G_c is simple and $SU(2)_c \subset G_c$.

to the $SU(2)_c$ for groups $SU(N)$, $SO(N)$ and $Sp(l)$ are investigated in § 3. It is found that only the fundamental representations of these groups can eventually be admissible. This result allows us to give the classification of the representations admissible with respect to the group $U(2)_c$. This is done in § 4. It is shown in § 5 that some fundamental physical requirements strongly restrict the class of admissible groups. It is found that as colour groups we can choose only the special unitary groups $SU(3r)$ where r is odd. It is shown that the total algebraic confinement of the $SU(3r)$ degrees of freedom holds.

2. Branching rules

In this section the following convention is adopted:

(λ) denotes the Young diagram of the irreducible representation $D(G_c)$ of the group G_c, i.e.



where $\lambda_i \geq \lambda_{i+1}$, $\lambda_i = 0$ for $i > l$, l is the rank of G_c and λ_i is the number of the Young boxes in the ith row;

$\Box$ is a Young box associated with the basic representation of G_c;

$\square$ is a Young box associated with the self-representation of $SU(2)_c$;

$\boxed{k}$ is a Young box associated with the kth self-representation of $SU(2)_c$ contained in the basic representation of G_c;

$\begin{array}{c}\boxed{p}\\ \square\end{array}$ is a Young diagram associated with pth scalar of $SU(2)_c$ contained in the basic representation of G_c;

d and s denote the number of $SU(2)_c$ doublets and singlets respectively contained in the basic representation of G_c.

We restrict ourselves to the case when the basic representation of G_c contains only the $SU(2)_c$ doublets and singlets, i.e.

$$\Box \downarrow SU(2)_c = \boxed{1} + \boxed{2} + \ldots + \boxed{d} + \begin{array}{c}\boxed{1}\\ \square\end{array} + \ldots + \begin{array}{c}\boxed{s}\\ \square\end{array} \tag{1}$$

and consequently

$$\dim \Box = 2d + s. \tag{2}$$

Following Hammermesh (1962) we can formulate the branching rules for reduction of the simple Lie groups to the $SU(2)_c$ in Young diagrams language. In every box of the Young diagram (λ) we write down the expansion (1) of $\Box \downarrow SU(2)_c$, i.e.

$$\Box \longrightarrow \boxed{\boxed{1} + \cdots + \boxed{d} + \begin{array}{c}\boxed{1}\\ \square\end{array} + \cdots + \begin{array}{c}\boxed{s}\\ \square\end{array}}$$

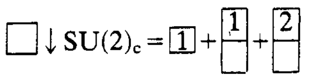
and then construct all possible SU(2) Young schemes following the rules (*i*)–(*iii*).

(*i*) From every box of the diagram (λ) we choose subdiagrams $\boxed{k}$ or $\begin{array}{|c|}\hline p\\\hline \ \\\hline\end{array}$ and multiply them according to the standard rules for SU(2). We repeat this procedure in all possible ways except the cases when in the same row or column of the basic diagram the indices of subdiagrams coincide.

(*ii*) In the later case we must take into account the symmetrisation (antisymmetrisation) of the basic boxes, i.e. in the constructed SU(2) Young schemes the subdiagrams ($\boxed{k}$ and $\boxed{k}$ or $\begin{array}{|c|}\hline p\\\hline \ \\\hline\end{array}$ and $\begin{array}{|c|}\hline p\\\hline \ \\\hline\end{array}$) must appear in the symmetric (antisymmetric) configuration.

(*iii*) We must take into account the transitivity of the symmetrisation (antisymmetrisation) relation in the rows and columns of the diagram (λ) to obtain the correct multiplicity of the SU(2) representations. More precisely, the $SU(2)_c$ diagrams obtained by a permutation of boxes in the rows or columns of the Young table (λ) are equivalent.

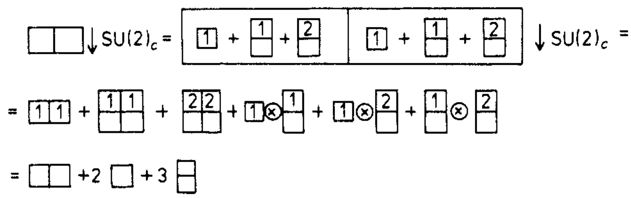
Examples. Let $G_c \sim SU(4)$ and



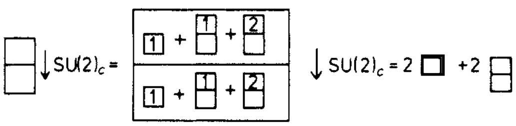
i.e.

$$\dim \square = 4,\ l = 3,\ d = 1,\ s = 2.$$

(*a*) for $(\lambda) = \square\square$ we have



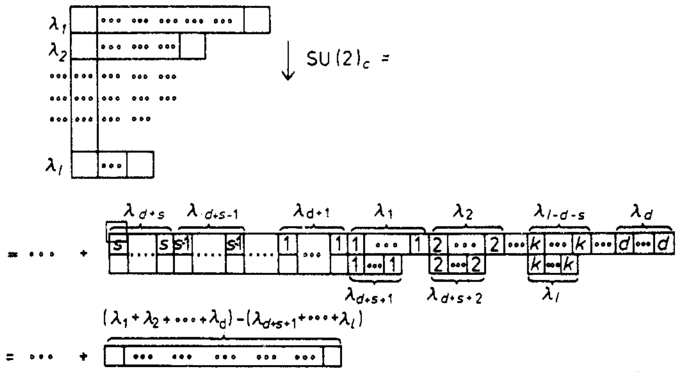
(*b*) for $(\lambda) = \begin{array}{|c|}\hline \ \\\hline \ \\\hline\end{array}$ we have



3. Representations admissible with respect to the $SU(2)_c$

A representation $D(G_c) \equiv (\lambda)$ of the group $G_c \supset SU(2)_c$ is admissible with respect to the $SU(2)_c$ iff it contains the $SU(2)_c$ singlets and doublets, that is only iff $(\lambda) \downarrow SU(2)_c = (\oplus \mathbf{1}) \oplus (\oplus \mathbf{2})$. It is obvious that a representation (λ) of G_c can be eventually admissible only if the basic representation $\square$ is admissible.

From the above definition and the branching rules (*i*)–(*iii*) it is easy to see that a representation (λ) is admissible iff in the expansion $(\lambda) \downarrow SU(2)_c$ the diagram with a maximal number of boxes in the first row is admissible:



Consequently the representation (λ) is admissible with respect to $SU(2)_c$ iff

$$(\lambda_1+\lambda_2+\ldots+\lambda_d)-(\lambda_{d+s+1}+\ldots+\lambda_l)=0 \tag{4a}$$

or

$$(\lambda_1+\lambda_2+\ldots+\lambda_d)-(\lambda_{d+s+1}+\ldots+\lambda_l)=1. \tag{4b}$$

We now apply this result to the determination of the admissible (with respect to $SU(2)_c$) representations for the groups $SU(N)$, $Sp(l)$ and $SO(N)$.

3.1. SU(*N*)

In this case $\dim \square = l+1 = 2d+s$, i.e. $d+s+1=l+2-d$. From equations (4*a*, *b*) we obtain

$$\lambda_1+(\lambda_2-\lambda_{l-d+2})+(\lambda_3-\lambda_{l-d+3})+\ldots+(\lambda_d-\lambda_l)=0 \text{ or } 1. \tag{5}$$

Because $\lambda_i \geqslant \lambda_{i+1}$ equation (5) admits only the following solutions:

(*a*) If $d>1$, $SU(N)$ singlet and self-representations are admissible ($\mathbf{1}$, $\boldsymbol{N}$ and $\bar{\boldsymbol{N}}$).

(*b*) If $d=1$, all fundamental representations and scalars are admissible ($\mathbf{1}$, $\boldsymbol{N}$, $\binom{N}{2}, \ldots, \binom{N}{r}, \ldots, \bar{\boldsymbol{N}}$).

3.2. Sp(*l*)

In this case $\dim \square = 2l = 2d+s$, i.e. $d+s+1=2l-d+1$. From equations (4*a*, *b*) we have (note that $0<d\leqslant l$)

$$\lambda_1+\lambda_2+\ldots+\lambda_d=0 \text{ or } 1. \tag{6}$$

Because $\lambda_i \geqslant \lambda_{i+1} \geqslant 0$, the group $Sp(l)$ possesses the following admissible representations:

(*a*) If $d>1$, scalar and self-representation ($\mathbf{1}$ and $\mathbf{2}\boldsymbol{l}$);

(*b*) If $d=1$, scalar and all fundamental representations.

3.3. SO(*N*) (*tensor representations*)

In the case $N=2l$, $\dim \square = 2l = 2d+s$, i.e. $d+s+1=2l-d+1$ and equations (4*a*, *b*) have the form (6). For $N=2l+1$, $\dim \square = 2l+1 = 2d+s$ i.e. $d+s+1=2l-d+2$. Because $0<d\leqslant l$, the equations (4*a*, *b*) have the form (6). Thus for $SO(N)$ the solution

is analogous to the Sp(l) case: admissible are

(a) scalar and self-representation if $d > 1$;

(b) scalar and all fundamental (tensor) representations if $d = 1$. However, the self-representation of SO(N) is real positive, while the SU(2) doublet is pseudo-real (Mehta 1966). Thus d is even and consequently case (b) must be ruled out.

The admissible spinor representations of SO(N) are considered in § 5.

4. Representations admissible with respect to the $U(2)_c$

A representation $D(G_c)$ of the group $G_c \supset U(2)_c$ is admissible with respect to the $U(2)_c$ if it is admissible with respect to the $SU(2)_c \subset U(2)_c$. Thus the results of the preceding section are sufficient to the determination of the representations of G_c admissible with respect to $U(2)_c$.

4.1. SU(N)

Because the self-representation of SU(N) is complex, d can be odd or even and consequently the representations admissible with respect to $U(2)_c$ and $SU(2)_c$ coincide.

4.2. Sp(l)

As is well known (see e.g. Mehta 1966) the self-representation **2*l*** of Sp(l) is real or pseudo-real. Thus the number d of the $U(2)_c$ doublets in **2*l*** must be even (with every doublet **2** must be associated the conjugate doublet $\bar{\mathbf{2}}$). Consequently $d > 1$ and from the discussion in the preceding section it follows that only singlet and self-representation of Sp(l) can be admissible. Note that the analogous result can be obtained if $Sp(l) \cap U(2)_c = SU(2)_c$ because in this case the centre of Sp(l) and $SU(2)_c$ must coincide i.e. $d = l$, so $d > 1$ for $l > 1$.

4.3. SO(N)

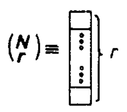
In this case the (tensor) representations are real. Consequently $d > 1$ and only the singlet and self-representation are admissible.

5. Physical limitations

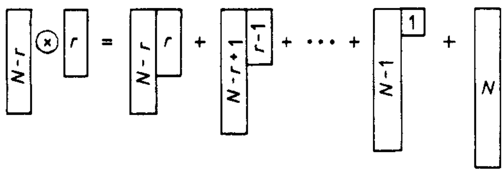
As is mentioned in § 1, in paper I it was shown that the observable states in QHS must necessarily be singlets of $SU(2)_c$. We now apply this condition to the admissible representations of the colour groups $G_c \supset SU(2)_c$.

4.4. SU(N)

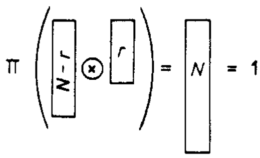
Let us assume for the moment that the $SU(2)_c$ singlets contained in the admissible representation



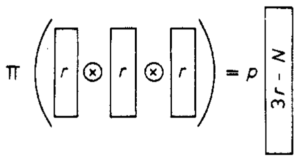
of SU(N) are associated with observable particles. Let us consider the particle–anti-particle states. Because



then



(as previously Π projects on the admissible representation space). Thus the particle–anti-particle state is the SU(N) singlet and is generally a mixture of the observable (singlets of $SU(2)_c$) and unobservable (doublets of $SU(2)_c$) one-particle states. Pure particle–anti-particle states exist only for $r = N$, i.e. the observable states are associated with the SU(N) singlets. A question arises as to the interpretation the unobservable admissible multiplets. To do this let us consider a three-particle state. From the multiplication rules for the Young diagrams we have (we can assume that $0 \neq r \leqslant N/2$)



Here $p = 0$ for $r < N/3$, $p = 1$ for $r = N/3$ etc., $3r - N \leqslant r$. If the representation $\binom{N}{r}$ is identified with the unobservable quark multiplet then the representation $\binom{N}{3r-N}$ must be associated with the (observable) baryon multiplet, i.e. with the SU(N) singlet. Consequently $N = 3r$. Moreover because the baryons are well described as quark bound-states with a symmetric (spatially and in flavour) wavefunction, Fermi statistics imply that they are antisymmetric with respect to the colour indices. Thus r must be odd.

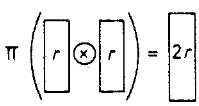
Concluding, in the SU(N) case

(a) the observable particles are associated with the SU(N) singlets, i.e. the colour degrees of freedom are confined;

(b) the unobservable quark multiplet is associated with the (admissible) representation $\binom{N}{r}$;

(c) only the groups SU($3r$), (i.e. $N = 3r$) where r is odd, are admissible.

Finally we note that the quark–antiquark (meson) states are observable whereas the diquark states are unobservable because



$2r = 2N/3 \neq 0$ or N

It is interesting that the quark multiplet is a self-representation only for $N=3$ $(r=1)$, i.e. for the colour group $G_c = SU(3)_c$.

4.5. Sp(l)

Let us assume for the moment that the admissible multiplet $\mathbf{2l}$ contains both observable ($SU(2)_c$ singlets) and unobservable ($SU(2)_c$ doublets) states. Because

$$\Pi(\square\otimes\square)=\Pi\left(\square\square+\text{—}+\mathbf{1}\right)=\mathbf{1},$$

the (admissible) singlet **1** contains a mixture of observable and unobservable states. Moreover this singlet is antisymmetric, i.e. the two-particle states cannot exist. Thus the observable states can be identified with Sp(l) singlets only. Furthermore, if the (unobservable) multiplet $\mathbf{2l}$ is associated with quarks then the baryon states are unobservable because

$$\Pi(\square\otimes\square\otimes\square)=3\square$$

i.e. the baryons form the unobservable multiplet $\mathbf{2l}$. For this reason the groups Sp(l) are rather inadequate to the description of the coloured states.

4.6. SO(N)

In the case of tensor representations considerations exactly analogous to the Sp(l) case imply that the observable states are SO(N) singlets, whereas the admissible self-representation $\boldsymbol{N}$ is unacceptable from the physical point of view.

Let us consider the spinor case. The product of the spinor representation D_S by D_S is direct sum of the tensor representations D_T i.e. $D_S\otimes D_S=\bigoplus_T D_T$. If D_S contains s (observable) $SU(2)_c$ scalars then $\bigoplus_T D_T$ must contain s^2 observable SO(N) singlets. But every SO(N) scalar which appears in $\bigoplus_T D_T$ is a mixture of the observable and unobservable states belonging to the D_S. Because $\bigoplus_T D_T$ necessarily contains the unacceptable (i.e. non-trivial) representations, from the irreducibility of D_S it follows that it is impossible to separate the s^2 (pure) observable states. Thus the spinor representations cannot contain observable particles. Furthermore because $D_S\otimes D_S\otimes D_S$ is the unobservable spinor representation then D_S cannot be associated with quark multiplet.

In conclusion the tensor and spinor representations of SO(N) (except for the trivial one) are unacceptable. Consequently SO(N) cannot be a colour group.

6. Conclusions

We have discussed the consequences of quaternionic structure of the Hilbert space for a symmetry of the theory. The admissible symmetry group must be of the form $G=G_F\times SU(3r)_{\text{colour}}$ where r is odd. The $SU(3r)_c$ degrees of freedom are algebraically confined. The unobservable quark multiplet is associated with the admissible representation $\binom{3r}{r}$. The group $SU(3)_{\text{colour}}$ $(r=1)$ is favoured for at least two reasons:

(*a*) Quarks are associated with the selfrepresentation (**3**) of $SU(3)_c$;

(*b*) Consequently this feature minimalises the number of colours; for example in the $SU(9)_c$ case ($r = 3$) the number of colours equals $\binom{9}{3} = 84$, and such a theory is more in the domain of science fiction.

Acknowledgments

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